Integral points in convex polyhedra, combinatorial Riemann-Roch Theorem and generalized Euler-MacLaurin formula

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The study of polyhedra in $\mathbb{R}^n$ with vertices on a given lattice is a very old subject.

It is a part of combinatorial geometry but has strong links to other fields: geometry of numbers, linear diophantine equations, algebraic geometry, for example.

In recent years, a *bridge* has been established between this subject and the theory of toric varieties [Kh-D].

For example, take the classical result of L. Ehrart: If Δ is a polyhedron in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, then the number of integral points in $p\Delta$ is a polynomial in p (for any positive integer p). An elegant proof of these results consists in passing the *bridge* we just mentioned and applying Riemann-Roch theorem for toric varieties.

In [D] Danilov asks for formulas connecting volumes of polyhedra and the number of integral points they contain.

To answer this question - and others! - the method of these notes is to forget the *bridge*, and use elementary methods in the study of polyhedra, or rather the space of linear combinations of characteristic functions of polyhedra. This space, first introduced in [G], has an interesting structure in itself (duality, convolution) which leads to natural generalizations of Ehrart's result as well as others, proved by McMullen, on *valuations*.

The second key-idea is to use integration of such functions as mentioned, with respect to the natural measure given by Euler characteristic. This idea has been developed by O.Viro in [V].

As a result of our study, we obtain a Riemann-Roch theorem in combinatorial geometry (Theorem 1, Chapter IV) which includes (passing the *bridge*) the classical Riemann-Roch theorem for toric varieties.

As a second result, we give a multidimensional generalization of the famous Euler-MacLaurin formula following the first "proof" by Euler.

The results of these notes belong to A. Khovanskii and A. Pukhlikov (some of them will appear in russian in Algebra and Analysis, 1992).

The first author wrote these notes after A. Khovanskii's lectures given in June 1991 at University Paris VII.

Paris, March 1992.

Content

Chapter I. The algebra of polyhedral chains: Euler integration, convolution, shadows.

We introduce, following an idea of Groemer [G], an algebra which contains canonically the set of convex compact polyhedra in $\mathbb{R}^n$ and finite union of such sets, and has an interesting structure of convolution algebra.

I. Polyhedral chains; convolution and duality.

I.1. In the classical theory of convex bodies, one considers compact convex polyhedra in $\mathbb{R}^n$, called in the following simply *polyhedra*, and the following operations on them [B-Z].

- Minkowski sum: For A and B polyhedra

$$A + B = \{x + y \; ; \; x \in A \;\; y \in B\}.$$

- Support function: For any linear form ξ on $\mathbb{R}^n$ and any polyhedron Δ the support function of Δ, L_Δ is defined by

$$L_\Delta(\xi) = \max_{x \in \Delta} \; \xi(x).$$

- Valuations on the set of polyhedra.

A function V with values in a group - usually the reals - defined on the set of polyhedra is called a *valuation* if it satisfies

$$V(\Delta_1 \cup \Delta_2) = V(\Delta_1) + V(\Delta_2) - V(\Delta_1 \cap \Delta_2)$$

whenever Δ_1, Δ_2 *and their union* are polyhedra.

A valued function V is *Minkowski additive* if

$$\begin{cases} V(\Delta_1 + \Delta_2) & = V(\Delta_1) + V(\Delta_2) \quad \text{for non-empty polyhedra} \\ V(\emptyset) & = 0. \end{cases}$$

Lemma 1. If V is Minkowski additive it is a valuation.

This results from the identity

$$\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 = \Delta_1 + \Delta_2$$

if Δ_1, Δ_2, $\Delta_1 \cup \Delta_2$ are non empty convex sets.

- The *dual* of a valuation V is the valuation V^*:

$$V^*(\Delta) = \sum (-1)^{|\Gamma|} \, V(\Gamma)$$

where the sum is taken over all faces Γ of Δ of all dimensions $|\Gamma|$.

I.2. The following generalization is introduced:

Definition 1. A *polyhedral chain* in $\mathbb{R}^n$ is a linear combination of characteristic functions of polyhedra.

$$f = \sum \lambda_i 1_{K_i} \qquad 1_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i \\ 0 & \text{if not.} \end{cases} \tag{1}$$

We denote by $\mathcal{P}$ the vector space of polyhedral chains. Any function f of $\mathcal{P}$ can be written in many ways as in (1). The set of polyhedra injects canonically in $\mathcal{P}$ by

$$K \mapsto 1_K.$$

Proposition 1 and Definition 2. If f is as in (1)

$$\int f \, dE = \sum \lambda_i$$

is independent of the representation of f. It is called the *Euler integral* of f. The Euler integral gives measure one to all characteristic functions of polyhedra.

Proof. Suppose

$$\sum \lambda_i 1_{K_i} = 0$$

and let us prove by induction that

$$\sum \lambda_i = 0.$$

a) If

$$n = 1$$

the K_i's are closed intervals of $\mathbb{R}$; we prove the assertion by induction on the number p of non-empty sets $K_i^0 \cap K_j^0$ (for i and j distinct).

If p is zero the result is clear: all K_i's have disjoint interiors, and it is easy to check that the λ_i must all vanish.

If p is strictly positive, let K_1 and K_2 be such that

$$K_1^0 \cap K_2^0 \neq \emptyset$$

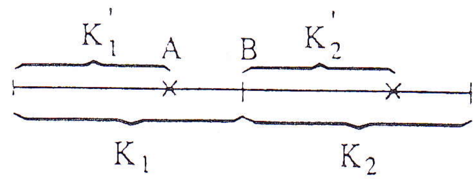

Figure 1

One has:

$$\lambda_1\, 1_{K_1} + \lambda_2\, 1_{K_2} = \lambda_1\, 1_{K_1'} + \lambda_2\, 1_{K_2'} + (\lambda_1 + \lambda_2) 1_{K_1 \cap K_2} - \lambda_1\, 1_A - \lambda_2\, 1_B$$

which shows that K_1, K_2 can be replaced by K_1', K_2', $K_1 \cap K_2$, A, B, thus lowering p by one. Whence the proof by induction.

b) Let π be a linear projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. It is enough to prove that

$$\sum \lambda_i\, 1_{K_i} = 0 \Rightarrow \sum \lambda_i\, 1_{\pi(K_i)} = 0.$$

Let x be a point in $\mathbb{R}^{n-1}$

$$\psi(x) = \sum \lambda_i\, 1_{\pi(K_i)}(x) = \sum_{i \in J} \lambda_i$$

where J is the set of indices i such that

$$x \in \pi(K_i) \Leftrightarrow \pi^{-1}(x) \cap K_i \neq \emptyset$$

$$\sum \lambda_i\, 1_{K_i} = 0 \Rightarrow \sum \lambda_i\, 1_{K_i \cap \pi^{-1}(x)} = 0$$

and

$$\sum \lambda_i\, 1_{K_i \cap \pi(x)} = \sum \lambda_i\, 1_{K_i \cap \pi^{-1}(x)} = 0 \Rightarrow \sum_{i \in J} \lambda_i = 0$$

because $\pi^{-1}(x)$ is of dimension one.

This proves that ψ is identically zero and so the sum is zero by induction.

Proposition 2 and Definition 3. If f and g are two polyhedral chains, the function

$$f * g = f * g(x) = \int f(x - y)\, g(y)\, dE(y)$$

is a polyhedral chain, called the *convolution* of f and g. With this convolution, $\mathcal{P}$ is a commutative algebra with unit (the Dirac mass δ_0), and the convolution prolongates Minkowski addition:

(2)
$$1_\Delta * 1_{\Delta'} = 1_{\Delta+\Delta'}.$$

Proof. For x fixed

$$f = \sum \lambda_i \, 1_{\Delta_i}$$

$$f(x-y) = \sum \lambda_i \, 1_{x-\Delta_i}$$

and one checks the formula (2); the convolution is well-defined, and makes $\mathcal{P}$ into a commutative algebra.

Remark. (2) is not true if Δ is not closed.

I.2. Duality.

Proposition 3 and Definition 4. Let f be a polyhedral chain. For any point x define

$$f^*(x) = \int_B f \, dE = \int f \cdot 1_B \, dE$$

where B is a sufficiently small polyhedral open ball centered in x. Then f^* is a well defined polyhedral chain, the *dual of* f. Moreover

(3)
$$f^{**} = f.$$

If

(4)
$$f = 1_\Delta \quad f^* = (-1)^{|\Delta|} \, 1_{\Delta^\circ}.$$

Proof. Take

$$f = 1_\Delta$$

with Δ polyhedron. Then, for B small open polyhedral ball centered at x_0

$$f^*(x_0) = \int 1_\Delta \cdot 1_B \, dE = \int \left(1_{\Delta\cap\overline{B}} - 1_{\Delta\cap B}\right) dE$$

is zero if x_0 is not in Δ^0. Otherwise

$$f^*(x_0) = 1 - \left(1 - (-1)^{|\Delta|}\right) = (-1)^{|\Delta|}.$$

One proves (3) the same way.

Proposition 4. If Δ is a polyhedron, 1_Δ is invertible in the algebra $\mathcal{P}$ and:

$$1_\Delta^{-1} = (-1)^{|\Delta|}\, 1_{-\Delta^0}.$$

Proof.

$$[1_\Delta * 1_{-\Delta^0}](x) = \int 1_\Delta(u)\, 1_{-\Delta^0}(x-u)\, dE(u).$$

If $x = 0$

$$1_\Delta * 1_{-\Delta^0}(0) = \int 1_\Delta(u)\, 1_{\Delta^0}(u) = \int 1_{\Delta_0}(u) = (-1)^{|\Delta|}.$$

If $x \neq 0$

$$1_\Delta * 1_{-\Delta^0}(x) = \int 1_\Delta(u)\, 1_{x+\Delta^0}(u) = \int 1_{\Delta \cap x+\Delta^0}(u)\, dE.$$

This Euler integral, being a topological invariant, can be computed easily: take for Δ a cube. Finally one gets (4).

I.3. Radon transformation. We define the Radon transform in real projective space, following ideas of O. Viro [V]. For this purpose, one needs an easy adaptation of previous discussion to the projective case. Rather, we prefer to change categories, and work in the category of *semialgebraic sets* in $\mathbb{RP}^n$.

Definition 5. A function f on $\mathbb{RP}^n$ is *constructible* if it is a linear combination of characteristic functions of closed semialgebraic sets A. As in the case of convex compacts polyhedra in $\mathbb{R}^n$, one can show that there exists a unique (finitely additive) measure on the space $\mathcal{F}$ of such functions, such that

$$E(1_A) = \int 1_A\, dE \qquad E \text{ Euler characteristic.}$$

Furthermore, this measure satisfies a Fubini-type property.

Definition 6. If f is in $\mathcal{F}$, the Radon transform of f is the function $\check{f}$ on $\mathbb{R}\check{\mathbb{P}}^n$ (set of hyperplanes of $\mathbb{RP}^n$) defined by

$$\check{f}(\ell) = \int_\ell f(x)\, dE(x) = \int f \cdot 1_\ell \, dE$$

ℓ hyperplane.

Theorem 5. [V] If n is even

$$f + f^{**} = \int f \, dE.$$

In all cases

$$\int \check{f} \, d\check{E} = \int f \, dE.$$

Application. Consider a generic two-dimensional smooth surface $\sum$ in $\mathbb{RP}^3$, and a generic projection on $\mathbb{R}^2$ (imbedded in $\mathbb{RP}^2$).

Theorem 6. For any x in $\mathbb{R}^2$ let

$a(x) = \text{card}\,(\pi^{-1}(x))$

$b(x) =$ number of tangent lines to the critical curve passing through x, *with orientation.*

Then

$$a(x) + b(x) = E(\textstyle\sum).$$

Proof. Let

$$f(x) = a(x)$$

$$\int f \, dE = E(\textstyle\sum).$$

Compute $f^*(\ell) = \int_{x \in \ell} a(x) = E\left(\pi^{-1}(\ell)\right)$, in general. The preimage of a generic line ℓ is a union of circles, then

$$f^*(\ell) = 0.$$

Only tangent lines count, each with a sign corresponding to the topology of $\pi^{-1}(\ell)$ and to the coorientation of the discriminant.

II. Shadows.

We introduce a new tool which will be useful in the study of general valuations.

Lemma 1. Let

$$I = [\alpha, \beta[$$
$$J =]\alpha, \delta]$$

be two half-intervals in $\mathbb{R}$, in opposite direction. Then

$$1_I * 1_J = 0.$$

Proof.

$$1_I * 1_J(x) = \int_{y+z=x} 1_I(y)\, 1_J(z)\, dE.$$

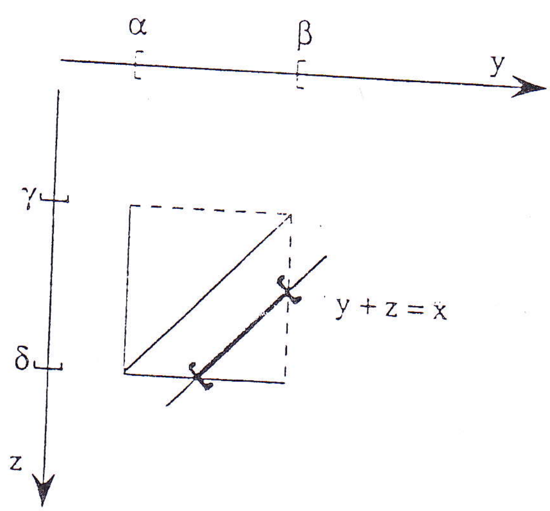

Figure 2

The line

$$y + z = x$$

intersects

$$\alpha \leq y < \beta$$
$$\gamma \leq z \leq \delta$$

in a half-closed interval.

Let now $\vec{a_1}, \ldots, \vec{a_k}$ be k vectors in $\mathbb{R}^n$, and denote by a_i the half intervals they define:

$$a_i = \{t\vec{a_i}\ ,\ 0 \leq t < 1\}$$

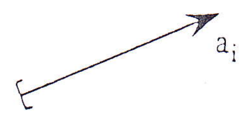

Figure 3

Lemma 2. Suppose a linear relation

$$\sum_i \lambda_i \vec{a_i} = \vec{0}$$

with (λ_i) all strictly positive. Then

$$1_{a_1} * 1_{a_2} * \cdots * 1_{a_k} = 0.$$

This confirms the existence of many zero divisors in $\mathcal{P}$.

Proof. Consider in the general case

$$f = 1_{a_1} * 1_{a_2}$$

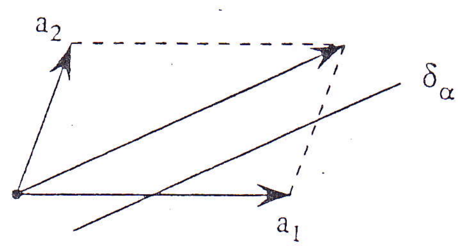

Figure 4

It is the characteristic function of the parallelogram built on $(\vec{a_1}, \vec{a_2})$ with two sides excluded. This can also be written, using the diagonal of this parallelogram and parallels to it (E_α measure on each parallel)

$$f = \int \left(\int f \mid_{\delta\alpha} dE_\alpha \right) dE(\alpha) \quad f_\alpha = \int f \mid_{\delta\alpha} dE_\alpha.$$

Repeating the same procedure for a_3

$$g = \int f_\alpha \, dE_\alpha * \chi_{a_3} = \int (f_\alpha * \chi_{a_3}) \, dE_\alpha$$

and so on. We get finally that

$$f = \chi_{a_1} * \cdots * \chi_{a_{k-1}}$$

can be written as a multiple integral of characteristic functions of half-intervals parallel to $(\vec{a_1} + \vec{a_2} + \cdots + \vec{a}_{k-1})$. Because of Lemma 1.

$$f * \chi_{a_k} = 0.$$

The remaining cases to be considered are those for which for some i

$$\begin{cases} \vec{a}_{i+1} = & \lambda(\vec{a_1} + \cdots + \vec{a_i}) \\ \vec{a}_{i+1} = & -\lambda(\vec{a_1} + \cdots + \vec{a_i}) \end{cases}$$

In the first case one can reduce to the case of $(k-1)$ vectors. and in the second case apply Lemma 1 to conclude.

Remark. This shows that $\mathcal{P}$ has a lot of zero divisors. It would be interesting to investigate further the structure of this convolution algebra.

Definition. If Δ is a polyhedron. and a a direction in $\mathbb{R}^n$ we call shadow of Δ with respect to a. and denote $S(\Delta.a)$ the part of the boundary of Δ which is "in *the shade when lighten from a*".

Precisely:

$$S(\Delta, a) = \{x \in \Delta : \exists u \in \mathbb{R}^n \, , \, x_1 \in \Delta \, , \, \Delta \cap \{u + ta \, . \, t \in \mathbb{R}\} = [x_1, x]\} \, .$$

Remark. $S(\Delta, a)$ is homeomorphic to the projection of Δ "parallel to a", in particular

$$\int S(\Delta.a) \, dE = 1.$$

Lemma 3. $S(\Delta, a)$ is a union of faces of Δ.

Notice first that $S(\Delta, a)$ is closed. Next, take x in the shadow $S(\Delta, a)$ and also in the relative interior of a face δ of Δ. Consider the affine space generated by δ and the line through x parallel to a. This space is divided in two half-spaces by δ, and points near x in δ are in the shadow. From this it results that the interior of δ is in the shadow, and also all of δ.

Lemma 4. Let Δ_1, Δ_2 be two polyhedra.

$$f_1 = 1_{\Delta_1} - 1_{S(\Delta_1,a)}$$
$$f_2 = 1_{\Delta_2} - 1_{S(\Delta_2,a)}$$

then

$$f_1 * f_2 = 0.$$

Lemma 5. Fubini's theorem. If

$$\pi : \mathbb{R}^n \to \mathbb{R}^m$$

is an affine map, and f a polyhedral chain in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(x)\, dE(x) = \int_{\mathbb{R}^m} \left\{ \int_{\pi^{-1}(y)} f(x)\, dE_y \right\} dE(y)$$

where dE_y is the Euler measure on the space $\pi^{-1}(y)$. The proof is elementary: take f to be the characteristic function of a polyhedron.

Take now f_1 and f_2 as in Lemma 4. From Lemma 5, applied with

$$\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$$

a linear projection parallel to a

$$f_1 * f_2(x) = \int \int_{x_1 \in \pi^{-1}(y)} f_1(x)\, f_2(x - x_1)\, dE_y\, dE(y).$$

Consider the first integral to perform: identifying $\pi^{-1}(y_1)$ with $\mathbb{R}$, dE_{y_1} with the Euler measure on $\mathbb{R}$, the restriction of f_1 gets identified with the characteristic function of a half-interval I and f_2 with the characteristic function of a half-interval J in opposite direction. Then the convolution is zero because of Lemma 1.

Chapter II. Valuations.

The study of valuations is popular: see [MH-S] for surveys. The previous results on convolution of polyhedral chains and shadows allow us to obtain general results on Λ-*polynomial valuations* for any lattice Λ in $\mathbb{R}^n$.

Definition 1. A valuation is a linear form on $\mathcal{P}$.

Theorem 1. [H2] Valuations on $\mathcal{P}$ identify with valuations on the space of polyhedra by

$$V(\Delta) = \widetilde{V}(1_\Delta).$$

The proof is technical.

It would be interesting to consider the semi-algebraic analogue.

Let Λ be a lattice in $\mathbb{R}^n$.

Definition 2. A Λ-polynomial valuation is a valuation V on $\mathcal{P}$ such that if

$$\tau_a(f)(x) = f(x+a)\ \tau_a.$$

Then

$$V(\tau_{-a}\, f) = (\tau_a\, V)(f)$$

is a polynomial in a for any a in Λ. If this polynomial is of degree k (for any non-zero a) V is said to be a Λ-*polynomial valuation of degree* k. For example a Λ-polynomial valuation of degree $(-\infty)$ is called a Λ-invariant valuation.

Remark. We do not suppose invariance by $GL_\Lambda(n)$. One essentially knows all Λ-invariant valuations with this property [B-K].

Examples.

1) $f \mapsto \int f\ dE$ is a Λ-invariant valuation (for any Λ).

2) Let Δ be a polyhedron, Q a polynomial in $\mathbb{R}^n$, then

$$V(\Delta) = \int_\Delta Q(x)\ dx$$

$$\widetilde{V}(\Delta) = \sum_{\Delta\cap\mathbb{Z}^n} Q(x)$$

define Λ-polynomial valuations.

General invariant Λ-valuations are defined by any additive measure on

$$X_\Lambda = \mathbb{R}^n/\Lambda.$$

II.1. From now on, we consider the ring

$$\mathcal{P} = \mathcal{P}_{\mathbb{Z}^n}$$

of polyhedral chains where the polyhedra Δ_i have vertices in $\mathbb{Z}^n$ (more generally, in lattice Λ). Convolution preserves $\mathcal{P}$.

Let J_0 be the set of polyhedral chains in $\mathcal{P}$ (or $\mathcal{P}_{\mathbb{Z}^n}$) with total zero mass

$$f \in J_0 : \int f \, dE = 0.$$

Lemma 1. J_0 is a maximal ideal.

Proof. The map

$$\int : \mathcal{P} \to \mathbb{R}$$

$$f \mapsto \int f \, dE$$

satisfies

$$\int f * g = \int f \int g$$

because of Lemma 5, Chapter II.

If V is a $\mathbb{Z}^n$-invariant valuation (called invariant valuation in the sequence)

$$V(f_a - f) = 0 \;\; \forall a \in \mathbb{Z}^n.$$

Let $\tilde{\mathcal{J}}_0$ denote the ideal of polyhedral chains killed by *any* invariant valuation, $\tilde{\mathcal{J}}_k$ the ideal of polyhedral chains killed by any $\mathbb{Z}^n$-polynomial invariant translation of degree up to k.

Theorem 2. a) $J_0^{n+k+1} \subset \tilde{\mathcal{J}}_k$

(convolution power for J_0!)

In particular,

b) $J_0^{n+1} \subset \tilde{\mathcal{J}}_0$.

Proof of Theorem. We will prove b), a) is a slight generalization of the argument.

Lemma 1. Let $f_1, \cdots, f_{n+1}$ be $n+1$ chains in J_0. Then

$$\varphi = f_1 * \cdots * f_{n+1}$$

can be written as a linear combination of convolutions of type

$$\delta_a * g$$

where a is a vertex of some fj.

Proof. 1) If

$$f = \sum \lambda_i 1_{K_i} \qquad \int f\, dE = 0 \Leftrightarrow \sum \lambda_i = 0$$

$f = \sum \lambda_i [1_{K_i} - \delta_0]$.

But

$$\int (1_{K_i} - \delta_0) = 0.$$

So it is enough to prove the lemma for such polyhedral chains.

2) Let $\mathcal{L}_i$ be an affine space of minimal dimension containing Δ_i and ν_i a vector in $\mathcal{L}_i$. One can choose

$$\nu_1, \ldots, \nu_{n+1} \quad \nu_i \in \mathcal{L}_i \subset \mathbb{R}^n$$

such that

$$\sum \lambda_i \nu_i = 0.$$

Take

$$\varphi = f_1 * \cdots * f_{n+1}$$

$$f_i = 1_{K_i} - 1_0 = \left(\underbrace{1_{K_i} - 1_{S(K_i, a_i)}}\right) + \left(\underbrace{1_{S(K_i, a_i)} - 1_0}\right)$$

g_i and h_i have zero total mass (see Remark on the shadow). Then

$$(*) \qquad \varphi = \Pi(g_i + h_i).$$

Because of Theorem 1, Chapter I, $\Pi g_i = 0$. All the other products in $(*)$ contain at least one function h_i whose support is of dimension strictly less than n (support of the f_i's).

The same procedure that allows us to pass from φ to a linear combination of functions of type

$$\psi = g_1 * \cdots * g_\ell * h_1 * \cdots * h_j$$
$$\ell + j = n + 1$$

(with at least one h) can be applied to ψ. Because $\mathcal{L}_i$ is choosen of minimal dimension, after a finite number of steps one is reduced to a combination of convolutions of functions each containing at least one function of type

$$h = \delta_a = 1_a - 1_0$$

where a is among the vertices of the f_i's: at each step no new vertex was added!

The part b) of Theorem 2) is an easy consequence of Lemma. Assuming a) true for $k = 1$, and letting V be a polynomial valuation of degree at most k, then for any integral point f

$$f \to V(\delta_b * f - \delta_0 * f) = V[(\delta_b - \delta_0) * f]$$

is a polynomial valuation of degree at most $(k-1)$. Applying Lemma 2 and the induction hypothesis yields part a) of Theorem 2.

Theorem 3. Let V be a polynomial valuation.
1) For any Δ in $\mathcal{P}$ and $k \in \mathbb{N}$

$$V\left(1_\Delta^{*k}\right) = V(k\Delta)$$

is a polynomial in k, $i_{V,\Delta}(k)$, called *Ehrart polynomial* for V.
2) $i_{V,\Delta}(-k) = V^*(k\Delta)$, k positif.
(Duality).
3) In general, for Δ_i polyhedra, γ a polynomial valuation

$$k_1, k_2, \ldots, k_\mu \to V(k_1\ \Delta_1 + k_2\ \Delta_2 + \cdots + k_\mu\ \Delta_\mu)$$

is a polynomial in $(k_1, \ldots, k_\mu)$ for

$$k_1 \geq 0 \cdots k_\mu \geq 0.$$

Theorem 3 extends classical results of McDonald and McMullen [MD,MM].

Proof. Suppose V is a polynomial valuation of degree at most p. We will prove first 1) and 2). The proof uses elementary algebra in the convolution structure of $\mathcal{P}$, and Theorem 2.

$$1_\Delta = 1 + (1_\Delta - 1)$$

with $1 = \delta_0$ unit element in $\mathcal{P}$.

For k positive, omitting the sign $*$ for convolution powers.

$$1_\Delta^k = [1 + (1_\Delta - 1)]^k$$

$$1_\Delta^k = \sum_{j=0}^{k} C_k^j (1_\Delta - 1)^j \ ; \quad \int (1_\Delta - 1)\, dE = 0.$$

Because of Theorem 2, only powers j intervene with

$$j < n + p + 1.$$

So we can write

$$V(1_\Delta^k) = \sum_{j=0}^{n+p} C_k^j \, V\,[(1_\Delta - 1)^j] \qquad (C_k^j = 0 \text{ if } j > k)$$

and this is a polynomial in k, denoted by $V_\Delta(k)$, of degree at most $n + p$.

II.2. Using the expansion of $(1 - k)^{-k}$ (k positive) we can write

$$1_\Delta^{-k} = \sum_{j=0}^{n+p} C_{-k}^j (1_\Delta - 1)^j + (1_\Delta - 1)^{n+p+1} * \varphi \ , \quad \varphi \in \mathcal{P}.$$

Because of Theorem 2 we deduce (remember the inverse of 1_Δ!)

$$\begin{aligned} V(1_\Delta^{-k}) = V_\Delta(-k) = V\left((-1)^{|\Delta|}\, 1^k_{-\Delta^0}\right) &= V^*(1^k_{-\Delta}) \\ &= V^*(k(1_{-\Delta})) \end{aligned}$$

which proves the assertion.

The third part of Theorem 3 is an easy extension of the first part.

Example. If V counts the number of points in $\mathbb{Z}^n \cap \Delta$. V^* counts - up to a sign - the number of points in $\mathbb{Z}^n \cap \Delta^0$. $V(k\Delta)$ and $V^*(k\Delta)$ are given up to a sign by the same polynomial taken for values k and $(-k)$.

Historical remarks.

The first author who studied systematically the number of integral points of integral polyhedra was L. Ehrart [E] who gave a generalization of Pick's formula. See also Reeve and McDonald [MD.R].

Then, in 1975, D. Bernstein gave a direct proof of the existence of Ehrart's polynomial

$$i_\Delta(n) = \text{card}(n\Delta \cap \mathbb{Z}^d)$$

for a polyhedra Δ in $\mathbb{R}^d$. His proof uses inductive methods on d and linear projections [B].

The theory of toric varieties allowed a reduction to Riemann-Roch theorem [K-D]. See [B] for complements and results close to some of the results herein.

Another progress was to replace the Ehrart's polynomial by another polynomial intrinsically defined through the generating function of the previous

$$(1-x)^{d+1} \sum i_{\Delta}(nP)x^n = h_0 + h_1 x + \cdots + h_d x^d.$$

This new polynomial is more practical, has local interpretation [B], [S], [MM2] and seems rich of further developments.

Still the coefficients of Ehrart's polynomial keep their mystery (see [K], [M], [R-G], [P] for the case of a tetrahedron). They are a basis for all unimodular continuous valuations on the set of $\mathbb{L}$-polyhedra [B-K].

Chapter III. Decomposition of polyhedra into cones and exponential sums.

We introduce cones in $\mathbb{R}^n$ and relate polyhedra to cones. This allows decomposition of sums over polyhedral chains.

III.1. Cones.

All cones we consider will be closed convex cones in $\mathbb{R}^n$, with vertex at any point. As with polyhedra, we introduce the

Definition 1. A *conic chain* in $\mathbb{R}^n$ is a linear combination of characteristic functions of cones. If C is a cone $\widetilde{C}$ denotes the cone translated from C, with vertex in 0.

Definitions 1'.

A cone with vertex at 0 is degenerate if it contains a non-zero vector subspace of $\mathbb{R}^n$,

$$C = E \times C'$$

E non-zero vector subspace.

A cone C with vertex at A is degenerate if $\widetilde{C}$ is degenerate.

A conic chain is degenerate if it can be represented as a linear combination of degenerate cones.

This induces an equivalence relation on the space of conic chains.

Decomposition of cones.

Let $f = \sum \lambda_i \, 1_{C_i}$ be a conic chain, all cones being convex and with vertex at 0.

Let L be a general linear form.

Proposition 1. There exists a conic chain f_1 equivalent to f,

$$f_1 = \sum \mu_j \, \chi_{D_j}$$

such that D_j are cones with vertex at 0, and L is strictly negative on D_j outside the origin.

Example. $n = 1$ $L(x) = x$. If $f = \chi_{\mathbb{R}-}$ take $f_1 = f$. If $f = \chi_{\mathbb{R}+}$ take $f_1 = -(\chi_{\mathbb{R}-} + \delta_0)$.

Proof. It is enough to consider for f the characteristic function of a cone. Consider the case $n = 2$ as an example.

a) If L keeps a constant sign on $C \backslash 0$ the assertion is obvious:

$$\begin{array}{llllll} f = \chi_C & L(x) < 0 & \forall x \in C\backslash 0 & f_1 & = & f \\ & L(x) > 0 & \forall x \in C\backslash 0 & f_1 & = & \chi_{C'} \\ & & & C' & = & -C\backslash\{0\} \end{array}$$

$$f + f_1 \sim 0.$$

b) If L does not keep a constant sign

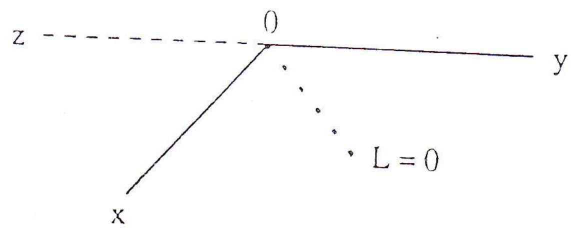

Figure 5

Suppose $C = \left(\widehat{x0y}\right)$, $L > 0$ on $0y$, $L < 0$ on $0x$. Take $C' = [z0x[$ ($0x$ excluded),

$$\chi_{C'} \equiv -\chi_C.$$

In the general case, let C be a simplicial cone (octant over a simplex), and suppose that L is sufficiently general so that it is not constant on any edge of dimension one of the simplex $(e_1, \dots, e_n)$.

Suppose $L(e_i) > 0$ on some e_i (otherwise take $f_1 = f$).

Replace C by C' cone over the simplex (e'_i) where

$$\begin{cases} e'_i = e_i & if \quad L(e_i) \leq 0 \\ e'_i = -e_i & if \quad L(e_i) > 0 \end{cases}$$

and take precisely C' to be the cone over the (non-closed) simplex S'

$$S' = \left\{ \sum_{L(e_i) \leq 0} \alpha_i \, e_i + \sum_{L(e_j) > 0} \alpha'_j \, e'_j \; ; \; \sum_{\substack{\alpha_i \geq 0 \\ \alpha'_j > 0}} \alpha_i + \alpha'_j = 1 \right\}.$$

It is clear that

$$L(x) < 0 \;\; \forall x \in C' \;, \; x \neq 0.$$

Moreover C and C' have no other intersection point than 0, and

$$f = \chi_{C''} + \delta_0 = \chi_C + \chi_{C'}$$

and C'' contains the whole line $z0y$.

III.2. Application to polyhedra.

Lemma 1. Suppose (C_i) are convex cones and L a linear form such that

$$L(x) \leq 0 \quad x \in C_i^* = C_i \backslash \{0\}.$$

Then

$$\sum \lambda_i \, 1_{C_i} \sim 0 \Rightarrow \sum \lambda_i \, 1_{C_i} = 0.$$

Let f be a polyhedral chain, A a point of its support. Suppose first f is the characteristic function of a polyhedron. Then there exists a unique cone $C_{\Delta,A}$ with vertex in A such that Δ and $C_{\Delta,A}$ have same germs at a. $C_{\Delta,A}$ is the tangent cone of Δ in a.

If A moves in the relative interior of a given face F of Δ

$$C_{\Delta,A} = C_{\Delta,F}$$

remains constant (up to a translation).

We deduce that for any linear combination f of characteristic functions of polyhedra and cones, and for any point A, there exists a conic chain γ such that, as germs in A

$$f \equiv \gamma \equiv \sum \lambda_i \, 1_{C_i}$$

where C_i have vertices in A.

Definition 2. If the C_i's can be choosen such that f is locally equal (in A) to a trivial chain γ, A is said to be a *trivial* point for f.

A *vertex* of f is a non-trivial point.

Examples. 1) If

$$f = 1_C \quad C \text{ cone with vertex in } A,$$

A is the only vertex of f (eventually, if C non-trivial; hopefully!).

2) If

$$f = 1_\Delta \ , \quad \Delta \text{ polyhedron},$$

the vertices of Δ are the vertices of f.

3) If

$$f = \sum \lambda_i 1_{\Delta_i}$$

the vertices of f are to be found among the vertices of the Δ_i.

Lemma 2. If f is a polyhedral chain, (A_i) its vertices, suppose

$$f = \sum_i \gamma_i = \sum_i \gamma'_i$$

with γ_i, γ'_i characteristic functions of cones with vertex A_i. Then

$$\gamma_i \sim \gamma'_i$$

$$\gamma''_i = \gamma_i - \gamma'_i = +\sum_{i \neq j} (\gamma'_i - \gamma_i)$$

implies that γ''_i coincides with a trivial conic chain in a neighbourhood of A and so everywhere by homogeneity.

The following result will be needed in the decompositions of polyhedral sums.

Theorem 1. (Varchenko) Let f be a polyhedral chain, $\mathcal{V}$ the finite set of its vertices, and L a general linear form.

There exists a family of conic chains $\{\gamma_A; A \in \mathcal{V}\}$ such that γ_A has vertex in A,

$$f = \sum \gamma_A$$

$$L(x) \leq L(A).$$

Proof. Take a general linear form L having a supremum at some vertex A of $\mathcal{V}$. The germ of f in A is equal to the germ of a conic chain γ_A. By Proposition 1 there exists γ'_A conic chain with vertex in A such that

$$L(x) \leq L(A) \quad \text{on all cones of } \gamma'_A$$

$$\gamma'_A \sim \gamma_A.$$

Consider

$$f_1 = f - \gamma'_A.$$

This new polyhedral chain has no other vertices than those of f and γ'_A, and is trivial near A. So f_1 has one vertex less than f. Furthermore L is bounded above on the support of f_1. By an easy induction argument, the theorem follows, because in the case of a chain f with no vertex, Lemma 1 implies that the chain f is zero.

Remark. All constructions above preserve the lattice structure. In particular if Δ is a polyhedron with vertices in $\mathbb{L}$ the γ's in Theorem 1 can also be chosen as combinations of characteristic functions of $\mathbb{L}$-cones.

III.3. Generalized discrete Gauss-Bonnet formula.

We will also need further the following result.

Proposition 2. Let

$$f = \sum \lambda_i \, 1_{K_i}$$

be a polyhedral chain, decomposed in cones as in Theorem 1. Then

$$\int f \, dE = \sum \lambda_i$$

and if $\widetilde{K_i}$ are the cones with vertex at 0 translated from the K_i's

$$\sum \lambda_i \, 1_{\widetilde{K_i}} = \left(\int f \, dE \right) \delta_0 .$$

Proof. 1) Take a convex polyhedron B big enough to contain the support of f. Then

$$\int f \, dE = \sum \lambda_i \int 1_{K_i \cap B} \, dE = \sum \lambda_i .$$

2) For $n = 1$, it is enough by linearity to prove the assertion for

$$f = 1_I \qquad I = [a, b]$$

then $f = \sum \lambda_i \, 1_{C_i}$ with cones C_i of three types:

$$C_i = [a_i, +\infty[\quad C_j =]-\infty, a_j] \quad C_k = \mathbb{R}$$

Then:

$$\sum \lambda_i + \sum \lambda_k = 0 \; . \; \sum \lambda_j + \sum \lambda_k = 0$$

$$\tilde{f} = \sum \lambda_i \, 1_{\widetilde{C_i}} = \left(\sum \lambda_i\right) 1_{[0,+\infty[} + \left(\sum \lambda_j\right) 1_{]-\infty,0]} + \left(\sum \lambda_k\right) 1_{\mathbb{R}} = \delta_0$$

because

$$1_{\mathbb{R}} = 1_{]-\infty,0]} + 1_{[0,+\infty[} - 1_0 .$$

3) Suppose the result of Proposition 2 true for all polyhedral chains in $\mathbb{R}^m$ and for all

$$m < n.$$

Intersect the given polyhedral chain f in $\mathbb{R}^n$ with any hyperplane H

$$f\,|_H = f \cdot 1_H = \sum \lambda_i\, 1_{C_i \cap H}$$

$$\sum \lambda_i\, 1_{C_i \cap H} = \left(\int f\,|_H \, dE_H \right) \delta_0 .$$

If f is a polyhedron Δ, for a given x in Δ choose an hyperplane H containing x

$$\sum \lambda_i\, 1_{\widetilde{K}_i}(x) = \sum \lambda_i\, 1_{\widetilde{K}_i \cap H}(x) = \left(\int f\,|_H \, dE_H \right) \delta_0(x)$$

$$= \delta_0(x).$$

This shows the assertion.

III.4. Exponential sums over cones.

1) Theorem 3. (Stanley) Let K be a rational cone with vertex at the origin, ℓ a linear form such that

$$\ell(x) < 0 \ \text{ in } \ K^*.$$

Then

$$S_K(p) = \sum_{x \in \mathbb{Z}^n \cap K} \exp\langle p, x \rangle$$

defined for p close to ℓ, is a meromorphic function of p in all $\mathbb{C}^n$.

Proof. 1) Take

$$n = 1 \ , \ \ K = \mathbb{R}_-$$

$$S_1(p) = \sum_{\substack{x \in \mathbb{N} \\ x \leq 0}} \exp px$$

Figure 6

converges for $p > 0$, and its sum equals

$$S_1(p) = \frac{1}{1 - \exp(-p)}.$$

2) In $\mathbb{R}^n$, if

$$K = \{\lambda e_1, \lambda \in \mathbb{R}\} \qquad e_1 \in 2^n$$

$$K \cap \mathbb{Z}^n = \mathbb{Z}_- e_1 = \{me_1 \in \mathbb{Z} \ , \ m \leq 0\}$$

$$\sum_{x \in \mathbb{Z}_- e_1} \exp\langle p, x\rangle = \frac{1}{1 - \exp(-pe_1)} \quad \text{if} \ \ \langle p, e_1\rangle > 0.$$

3) Suppose K is simple and integer simple, and

$$K = \left\{ \sum_{i=1}^{p} \lambda_i \ e_i \ ; \ \lambda_i \geq 0 \right\}$$

where e_i are fixed integral vertices which generate freely a sublattice of $\mathbb{Z}^n$.

Let

$$Q = \left\{ \sum_{i=1}^{p} r_i \ e_i \ \ 0 \leq r_i < 1 \right\}.$$

Then Q contains a certain number of points in $\mathbb{Z}^n_+$, and any point x in $\mathbb{Z}^n_+$ can be written in a unique way

$$x = u + v \quad \begin{array}{rcl} u & \in & K \\ v & \in & Q \cap \mathbb{Z}^n_+. \end{array}$$

Adding the various contributions for each v

(1) $$S_K(p) = \frac{1}{\sum_{y \in Q} \exp py} \prod_{i=1}^{n} (1 - \exp(-pe_i))$$

which is meromorphic.

In general, K can be decomposed into simple cones, and the sum $S_K(p)$ can be written as a sum of terms like (1), a meromorphic function in the whole space (by analytic continuation).

Let $\mathcal{C}_0$ be the space of linear combinations of characteristic functions of rational cones with vertex in 0.

Proposition 4. There is a unique extension of

$$K \to S_K(p)$$

to $\mathcal{C}_0$:

$$f \mapsto S_f(p)$$

which associates to any f in $\mathcal{C}_0$ a meromorphic function $S_f(p)$, such that if f is trivial $S_f(p)$ is zero.

Proof. To define S_f take a general linear form ℓ, and g equivalent to f as in Theorem 1.

Then define

$$S_f(p) = S_g(p)$$

as a meromorphic function, holomorphic for p close to ℓ. It does not depend on the choice of g, because of Lemma 1.

Theorem 5. Let Δ be an (integral) polyhedron, $\mathcal{V}$ its set of vertices

$$1_\Delta = \sum_{A \in \mathcal{V}} \gamma_A$$

a decomposition of Δ in rational conic chains with vertices in $\mathcal{V}$, as in Theorem 1, corresponding to a given general linear form ℓ. Then

1)
$$\sum_{x \in \Delta \cap \mathbb{Z}^n} \exp px = \sum_{A \in \mathcal{V}} \exp pA \; S_{\widetilde{\gamma_A}}(p)$$

where $\widetilde{\gamma_A}$ is the conic chain with vertex in 0 obtained by translation from γ_A.

2)
$$\sum_{A \in \mathcal{V}} S_{\widetilde{\gamma_A}}(p) \equiv 1.$$

Proof. a) The first part, for p close to ℓ, is a consequence of Theorem 1 because

$$S_{\gamma_A}(p) = \exp pA \; S_{\widetilde{\gamma_A}}(p).$$

b) We can apply Proposition 2 (Gauss-Bonnet).

2) Continuous analogues

Proposition 6. Let K be any rational cone in $\mathbb{R}^n$

$$M_K(p) = \int \exp(px)dx$$

is convergent for p such that

$$px = \langle p, x\rangle \quad < 0 \quad x \in K \quad x \neq 0.$$

This integral defines a *rational* function of p.

Proof. a) $n = 1 \quad p > 0 \quad K = \mathbb{R}_-$

$$\int_{-\infty}^{0} \exp \; px \; dx = \frac{1}{p}.$$

The formal sum - defined for no p! - should be zero:

$$\int_{\mathbb{R}} \exp px \; dx = \int_0^{+\infty} \exp px + \int_{-\infty}^{0} \exp px = \frac{-1}{p} + \frac{1}{p} = 0.$$

b) If K is a simplicial cone of dimension n,

$$K = \left\{ \sum_1^n \lambda_i \, e_i \; ; \; \lambda_i \geq 0 \right\}.$$

Consider a linear change of variable $\mathcal{A}$ which changes the canonical simplicial cone Φ into K

$$\int_K \exp(px)dx = \int_\Phi \exp(p\mathcal{A}y)|\det \mathcal{A}|dy = \int_\Phi \exp \sum p_i \; y_i \; |\det \mathcal{A}| \; dy$$

$$= \frac{\det(e_i, \ldots, e_n)}{p_1 \cdots p_n}$$

$$p_i = \langle p, e_i\rangle$$

which is a rational function.

c) For a general rational cone, the construction is similar to the one in the discrete case.

Geometrical interpretation. If $V(p)$ is the volume of the intersection of K with the half-space

$$px \geq -1$$

(2)
$$V(p) = \frac{1}{n!} \int_K \exp px \, dx.$$

In fact if $V(p, \alpha)$ is the $(n-1)$-volume of the intersection of K with the hyperplane

$$px = \alpha \qquad \alpha < 0$$

one has

$$V(p, \alpha) = |\alpha|^{n-1} \, V(p)$$

and also

$$\int e^{\alpha} \, |\alpha|^{n-1} \, V(p) d\alpha = \int_K \exp px \, dx.$$

Theorem 7. If Δ is any polyhedron, $\mathcal{V}$ the set of its vertices, $\widetilde{\gamma_A}$ the cone with vertex in 0 translated from the tangent cone γ_A of Δ in A,

1)
$$\int_{\Delta} \exp px \, dx = \sum_{A \in \mathcal{V}} \exp pA \; M_A(p)$$

where

$$M_A(p) = M_{\widetilde{\gamma_A}}(p)$$

2)
$$\sum_{A \in \mathcal{V}} M_A(p) \equiv 0.$$

The proof follows the line of the proof of Theorem, using the decomposition of Δ into conic chains (Theorem 1) and Gauss-Bonnet.

Remarks. 1) Both Theorems 5 and 7 have an obvious extension to polyhedral chains. For example

$$\sum_{x \in \mathbb{Z}^n} \exp px f(x) = \sum_{A \in \mathcal{V}} \exp pA \; f(A) \; M_A(p).$$

2) For a given cone C the formula

$$\sum_{x \in \Delta \cap \mathbb{Z}^n} \exp px = \sum \exp pA \; S_A(p)$$

is a *universal* formula: the measures $S_A(p)$ are defined by the tangent cone of Δ at A and the formula needs only, to be applied, a decomposition of each tangent cone into simple and integer simple cones to get simple formulas for $S_A(p)$.

For a given fixed

$$p = (p_1, \ldots, p_n)$$

take

$$\lambda p = (\lambda p_1, \ldots, \lambda p_n).$$

The left-hand side is a holomorphic function and its value for zero is $\#(\mathbb{Z}^n \cap \Delta)$.

The right-hand side is a meromorphic function, which has a Laurent series at the origin

$$\sum_{i \geq i_0} A_i \, (P) \lambda^i.$$

The A_i's depend on the decomposition of Δ in cones (on a choice of any privilegied direction). For different directions p one gets different decompositions. The choice of p allows the choice of decay-functions corresponding to cut-offs for the integrals associated to the hyperplane δ going to infinity.

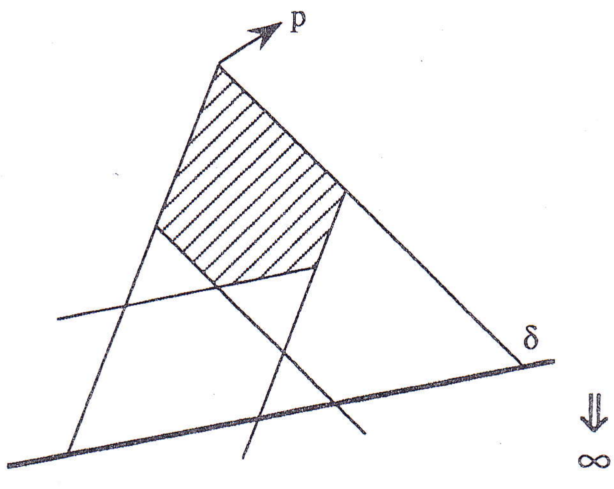

Figure 7

Remark. 3) By differentiation with respect to p the formulae extends to a summation formula for example for

$$E = \sum_{x \in \Delta \cap \mathbb{Z}^n} Q(x) \exp px$$

$Q(x)$ polynomial.

Chapter IV. Todd operators and generalized Euler-MacLaurin formulae.

Inspired by Euler's original proof we introduce a non-local infinite order differential operator, called Todd operator because of its formal relation to the Todd class. This construction allows the proof of a relation between discrete and continuous valuations, called "*combinatorial Riemann-Roch theorem*" (as a particular case one gets Riemann-Roch theorem for non-singular toric varieties).

We also obtain natural generalizations of the classical asymptotic formula of Euler-MacLaurin.

I. Operators of infinite order.

I.1. It is well known that if U is an open set in $\mathbb{C}^n$, and (a_α) holomorphic functions in U, the infinite series

$$P\left(x, \frac{\partial}{\partial x}\right) = \sum_\alpha a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$$

(with usual conventions) defines a local operator, which acts on holomorphic functions in U and on the sheaf $\mathcal{O}_U$ if strict convergence conditions are satisfied [B-K].

In the general case, take for example

$$P\left(x, \frac{\partial}{\partial x}\right) = \sum a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha \quad \text{where} \quad a_\alpha \in \mathcal{O}(|z| < \varepsilon).$$

Suppose for some R in $\mathbb{R}^n_+$,

$$\sum |a_\alpha(x)| \; R^\alpha < \infty$$

then it is easy to see that P acts on the space $\mathbb{E}_R$ of exponential-polynoms of the type

$$\varphi \in \mathbb{E}_R \quad \varphi = \sum_{\text{finite}} \exp(p_j x) \; q_j(x)$$

$$\begin{cases} q_j(x) \in \mathbb{R}[x] \\ |p_j| < R \end{cases}$$

Moreover:

(1) $$P\left(x, \frac{\partial}{\partial x}\right) [\exp px] = P(x,p) \exp(px).$$

Basic example. The Todd operator in n variables is the non-local infinite order differential operator associated with the function

$$\Pi(\xi) = \prod_{i=1}^{n} \frac{\xi_i}{1 - \exp(-\xi_i)}$$

that is, take the convergent series associated with each term of the product, and replace ξ_i by $\frac{\partial}{\partial x_i}$.

II. Framed polyhedra.

Definition 1. A *framed polyhedron* in $\mathbb{R}^n$ is the set of following data:

a) A *simple* polyhedron Δ: at each vertex of Δ, Δ is the intersection of n half-spaces.

b) For each face F of Δ, a dual element ℓ_i in $\mathbb{R}^{n*}$.

c) A volume form ω_n.

Basic example. Take any lattice $\mathbb{L}$ in $\mathbb{R}^n$, Δ a simple polyhedron with vertices in $\mathbb{L}$. Then associated to it is a canonical framed polyhedron choose for any face F_i of Δ the unique ℓ_i defined by

$\ell_i \perp F_i$

$\ell_i \leq d_i$ on Δ,

$\Delta = \{x \; ; \; \ell_i(x) \leq d_i \; ; \; i = 1, \ldots, j\}$

$\ell_i \in \mathbb{L}^*$,

with smallest length (with respect to $\mathbb{L}$) for these properties.

Take for ω_n the volume associated to $\mathbb{L}$.

To such a framed polyhedron, we associate a *family of polyhedra* in $\mathbb{R}^n$, depending on parameters (h); these polyhedra have no more vertices; they are obtained by moving the faces F_i parallel to themselves:

$$\begin{aligned} \Delta_h &= \{x; \ell_i(x) \leq d_i + h_i \quad i = 1, \ldots, j\} \\ h &= (h_1, \ldots, h_j) \\ \Delta_0 &= \Delta \end{aligned}$$

Theorem 1. Combinatorial Riemann-Roch theorem. Suppose Δ is simple and integral simple, and let

$$g(h) = \int_{\Delta_h} Q(x)\, dx$$

where Q is a given function in $\mathbb{R}^n$.

Let $\mathbb{T}(\partial/\partial h)$ be the Todd operator in the j variables h. Then, for Q polynomial,

$$\mathbb{T}(\partial/\partial h)[g(h)] \mid_{h=0} = \sum_{x \in \Delta \cap \mathbb{Z}^n} Q(x).$$

This formula is also true for Q in $\mathbb{E}_R$ (for small R).

We assume Δ as in hypothesis.

Lemma 1. The combinatorial structure of Δ_h is independent of h for small h and

$$g(h,p) = \int_{\Delta_h} \exp(px)dx.$$

From the hypothesis we deduce that each vertex A of Δ is defined by the intersection of n hyperplanes

$$\ell_i(x) = d_i$$

indexed by a subset $I(A)$ of $\{1, \dots, j\}$.

For h close to zero the n corresponding equations

$$\ell_i(x) = d_i + h_i \tag{1}$$

define the vertex $A(h)$ of Δ_h. In the same way any face of codimension q of $\Delta(h)$ is defined by q equations as (1). This shows that the combinatorial structure does not change for h close to zero.

Proof of Theorem 1. From Theorem 7 we know:

$$\int_\Delta \exp px \; dx = \sum_{A \in \mathcal{V}} \exp(pA) \; M_A(p) \tag{2}$$

where $\mathcal{V}$ is the set of vertices of Δ, the equality being between meromorphic functions.

Now (2) is still true for Δ_h !

$$\int_{\Delta_h} \exp px \; dx = \sum_{A \in \mathcal{V}(\Delta_h)} \exp(pA) \; M_{A_h}(p) \tag{3}$$

where $\mathcal{V}(\Delta_h)$ denotes the set of vertices of Δ_h.

If Δ_h has rational vertices (in $\frac{1}{m}\mathbb{L}$), take $m\Delta_h$ and apply (3). Because (3) is smooth in h, this extends to any h.

Given a vertex A_h of Δ_h, because of Lemma 1 A_h can be defined by n hyperplanes, Δ being simple and integral simple these hyperplanes can be taken as hyperplanes of coordinates:

$$\ell_i(x) = x_i = d_i + h_i \qquad i = 1, \dots, n.$$

Then the contribution of A_h to the second term of (3) is

$$C_{A_h} = \exp\left(\sum p_i\, \widetilde{x}_i\right) \times \frac{1}{\prod p_j}$$
$$p_j = \langle p, e_j \rangle$$
$$A_h = (\widetilde{x}_i)$$
$$C_{A_h} = \exp\left(\sum_{i=1}^{n} p_i(d_i + h_i)\right) \frac{1}{\prod_{i=1}^{n} p_i}.$$

Applying the Todd operator to this function of h we get

$$\mathbb{T}(h)[C_{A_h}] = \frac{\prod_1^n p_j}{\prod_1^n \exp(-p_j)} \frac{1}{\prod p_j} \exp \sum_1^n p_i(d_i + h_i)$$
$$\mathbb{T}(h)[C_{A_h}] = \exp p\ A(h) \times S_{\widetilde{\gamma_A}}(p).$$

Finally, summing over all vertices of Δ_h, and letting h equal zero:

$$\mathbb{T}(h)[C_{A_h}]\,|_{h=0} = \sum \exp p\ A\ S_{\gamma_A}(p) = \sum_{x \in \Delta \cap \mathbb{Z}^n} \exp(px)$$

which is valid for p small, because of Theorem 5, Chapter III.

This proves Theorem 1 for

$$Q(x) = \exp(px)$$

p small.

If now $R(\partial/\partial p)$ is any differential operator with constant coefficients

$$R(\partial/\partial p) \exp(px) = R(x) \exp(px)$$

(4)
$$R(\partial/\partial p)\left[\mathbb{T}(\partial/\partial h) \int_{\Delta_h} \exp(px)dx\right]\Big|_{h=0} = \mathbb{T}(h) \int_{\Delta_h} R(x)\exp(px)dx\,|_{h=0} = \sum_{x \in \Delta \cap \mathbb{Z}^n} R(x)\exp(px).$$

When p goes to zero the equality (4) becomes

$$\mathbb{T}(\partial/\partial h)\int_{\Delta_h} R(x)dx\,|_{h=0}= \sum_{x\in\Delta\cap\mathbb{Z}^n} R(x).$$

This proves Theorem 1. In particular one gets a new formula for the number of points of $\mathbb{Z}^n\cap\Delta$:

Corollary. For Δ as above

$$\mathbb{T}(\partial/\partial h)[\mathrm{Vol}(\Delta_h)]\,|_{h=0} = \mathrm{card}(\Delta\cap\mathbb{Z}^n).$$

Application. Euler-MacLaurin formula.

In the one dimensional case the theorem gives an asymptotic formula of Euler-MacLaurin, in case of annihilation of the remainder term.

For various choices of $Q(x)$ Theorem 1 can be considered as a systematic generalization of Euler-MacLaurin formula in higher dimensions.

We hope to return to these questions soon.

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